LECTURE 8

Chapter 2 Review

Moral of continuity. If a function is continuous at a point $x = a$, then

$$\lim_{x\to a} f(x) = f(a)$$

which means you can plug in. You only worry about the existence of the limit when a is NOT a continuity point of f, e.g. at jumps, vertical asymptotes, etc.

On the other hand, to **show** a function is continuous at a point $x = a$, you check

(1) Existence of left limit $\lim_{x\to a^-} f(x)$ and if exists, find its value.
(2) Existence of right limit $\lim_{x\to a^+} f(x)$, and if exists, find its value.
(3) The function value $f(a)$ at the point of interest.
(4) See if step 1,2,3 give the same number.

Nice example where step 1 and 2 are the same but not step 3. Consider a function with a hole, and takes value elsewhere.

$$f(x) = \begin{cases} 1, & x = 2, \\ 3, & x \neq 2. \end{cases}$$

Section 3.1 Tangent Lines and the Derivative at a Point

Rise over run expressed by function values. Given $[a, b]$, the slope of the secant line m at $x = a$ and $x = b$ is

$$m = \frac{f(b) - f(a)}{b - a}.$$

Now, imagine we let b approach a in an infinitesimal manner, say, $b = a + h$ for some number h we can continuously toggle. Then the above slope becomes

$$m = \frac{f(a+h) - f(a)}{a + h - a} = \frac{f(a+h) - f(a)}{h}$$

which is often referred to as the **average rate of change**. If we make h very small such that $x = a + h$ and $x = a$ eventually collide, we arrive at a tangent line instead of a secant line, which has slope

$$f'(a) := \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

AS A DEFINITION of the derivative of f at $x = a$, given that the limit exists. This is also called the **instantaneous rate of change**.

Remark. One always speaks of **average rate of change** for function values taken over an interval. It is trivial to speak of it at a particular point because the average of one point is itself. One always speaks of **instantaneous rate of change** for a specified point, as the point of tagency is unique.

Example 1. Find the slope of the function's graph at a given point and then determine the equation of the tangent line. (Typical exam question)

(1) $f(x) = x^2 + 1, \quad (2, 5).$

Solution. Consider the difference quotient, evaluated at $x = 2$.

$$\begin{aligned}\lim_{h\to 0} \frac{f(2+h) - f(2)}{h} &= \lim_{h\to 0} \frac{(2+h)^2 + 1 - \left(2^2+1\right)}{h} \\ &= \lim_{h\to 0} \frac{h^2 + 2h + 4 + 1 - 5}{h} \\ &= \lim_{h\to 0} \frac{h^2 + 2h}{h} \\ &= \lim_{h\to 0} h + 2 \\ &= 2\end{aligned}$$

Then point-slope form yields,

$$y = y_0 + m(x - x_0) = 5 + 2(x - 2) = 2x + 1.$$

(2) $g(x) = \sqrt{x+1}, \quad (3, 2)$.

Solution. Consider the difference quotient evaluated at $x = 3$.

$$\begin{aligned}\lim_{h\to 0} \frac{f(3+h) - f(3)}{h} &= \lim_{h\to 0} \frac{\sqrt{3+h+1} - \sqrt{3+1}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{4+h} - 2}{h} \\ &= \lim_{h\to 0} \frac{\left(\sqrt{4+h} - 2\right)\left(\sqrt{4+h} + 2\right)}{h\left(\sqrt{4+h} + 2\right)} \\ &= \lim_{h\to 0} \frac{4 + h - 4}{h\left(\sqrt{4+h} + 2\right)} \\ &= \lim_{h\to 0} \frac{1}{\sqrt{4+h} + 2} \\ &= \frac{1}{4}\end{aligned}$$

Then point-slope form yields,

$$y = y_0 + m(x - x_0) = 2 + \frac{1}{4}(x - 3) = \frac{1}{4}x + \frac{5}{4}.$$

Thought process: what does the question want? A slope first – so do the limit with difference quotient. Then, once the slope is determined, the tangent line also shares the slope. With a known point and the slope, the equation of the tangent line can be decided.

Example 2. Find the slope of the curve at the point indicated.

$$y = \frac{x-1}{x+1}, \quad x = 0.$$

Solution. Difference quotient evaluated at $x = 0$.

$$\begin{aligned}\lim_{h\to 0} \frac{y(h) - y(0)}{h} &= \lim_{h\to 0} \frac{\frac{h-1}{h+1} - \left(\frac{-1}{1}\right)}{h} \\ &= \lim_{h\to 0} \frac{\frac{h-1}{h+1} + \frac{h+1}{h+1}}{h} \\ &= \lim_{h\to 0} \frac{2h}{h(h+1)} \\ &= \lim_{h\to 0} \frac{2}{h+1} \\ &= 2\end{aligned}$$

Section 3.2 The Derivative as a Function

This idea is similar to that of a continuous function. You can check whether a function is continuous at a point by computing the left and right limit and seeing if they match and are equal to the function value. A continuous function on an interval then means it is continuous at every point on this interval. Similarly, we say a function is differentiable at a point by considering the limit of the difference quotient, checking its existence first, and then computing its value, that is, the derivative at this point. A differentiable function on an interval means it is differentiable at every point on this interval except at the endpoints. It is said to have a derivative as a function of the independent variable $x \in (a, b)$, such that

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}, \quad x \in (a, b),$$

or with an alternative definition

$$f'(x) = \lim_{z\to x} \frac{f(z) - f(x)}{z - x}, \quad x \in (a, b).$$

Last section, we find the derivative at **a particular** point. The idea in this section is to find a general form of the derivative at **any** given point.

Example 3. Find $\frac{dp}{dq}$ where $p = q^{\frac{3}{2}}$.

$$\begin{aligned}
p'(q) &= \lim_{h\to 0} \frac{f(q+h) - f(q)}{h} \\
&= \lim_{h\to 0} \frac{(q+h)^{\frac{3}{2}} - q^{\frac{3}{2}}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{(q+h)^3} - \sqrt{q^3}}{h} \\
&= \lim_{h\to 0} \frac{\left(\sqrt{(q+h)^3} - \sqrt{q^3}\right)\left(\sqrt{(q+h)^3} + \sqrt{q^3}\right)}{h\left(\sqrt{(q+h)^3} + \sqrt{q^3}\right)} \\
&= \lim_{h\to 0} \frac{(q+h)^3 - q^3}{h\left(\sqrt{(q+h)^3} + \sqrt{q^3}\right)} \\
&= \lim_{h\to 0} \frac{q^3 + 3q^2h + 3qh^2 + h^3 - q^3}{h\left(\sqrt{(q+h)^3} + \sqrt{q^3}\right)} \\
&= \lim_{h\to 0} \frac{3q^2 + 3qh + h^2}{\sqrt{(q+h)^3} + \sqrt{q^3}} \\
&= \frac{3q^2}{2q^{\frac{3}{2}}} \\
&= \frac{3}{2} q^{\frac{1}{2}}
\end{aligned}$$